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Maximally entangled mixed states of two atoms trapped inside an optical cavity

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In some off-resonant cases, the reduced density matrix of two atoms symmetrically coupled with an optical cavity can very approximately approach to maximally entangled mixed states or maximal Bell violation mixed states in their evolution. The influence of phase decoherence on the generation of maximally entangled mixed state is also discussed.

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Quantum entanglement plays a crucial role in quantum information processes [1]. In the past few years, much attention has been paid to the preparation of maximally entangled mixed states [2, 3, 4]. The properties of maximally entangled mixed state have been studied by many authors [5, 6, 7]. Maximally entangled mixed states are those states that, for a given mixedness, achieve the greatest possible entanglement. For two-qubit systems and for various combinations of entanglement and mixedness measures, the forms of the corresponding maximally entangled mixed states are different [7]. Using correlated photons from parametric down-conversion, maximally entangled mixed states in the linear entropy-concurrence plane have been created and characterized [2]. Generation and characterization of two-photon polarization maximally entangled mixed states in the linear entropy-concurrence plane has been also carried out, which is based on the peculiar spatial characteristics of a high brilliance source of entangled pairs [3]. The preparation of maximally entangled mixed states of two atoms asymmetrically on-resonance coupled with an optical cavity has also been proposed [4]. Recently, Clark and Parkins [8] have proposed a scheme to controllably entangle the internal states of two atoms trapped in a high-finesse optical cavity by employing quantum-reservoir engineering. By applying the on-resonance atom-cavity couplings which are time-dependent, Olaya-Castro et al. have also presented an efficient scheme for controlled generation of entangled states of two atoms inside an optical cavity[9]. However, truly resonant coupling is not available in realistic physical system. It is desirable to investigate how the off-resonance coupling affects the preparation of maximally entangled mixed states of two atoms. Based on our previous analytical results in Ref.[10], in which the entanglement behaviors of two atoms inside an optical cavity in the presence of phase decoherence have been derived, we can easily analyze the feasibility for preparing maximally entangled mixed states in such a system. For keep this paper self-contained, we briefly outline the basic contents about two two-level atoms inside an optical cavity. Here, we investigate two two-level atoms symmetrically coupling to single mode optical cavity and show that in some off-resonant cases, the maximally entangled mixed states in the plane of concurrence versus linear entropy of two atoms can be very approximately generated. It is shown that the long time entanglement behavior of two atoms is sensitive to the ratio of the detuning and the coupling strength. The influence of the initial mixedness of the atoms and phase decoherence is also analyzed.

Considering the system that two atoms are trapped inside single mode optical cavity initially prepared in the vacuum state. The Hamiltonian for the system can be given by [11, 12] ($\hbar = 1$),

$$H = \frac{\omega_0}{2} \sum_i^2 \sigma_z^{(i)} + \omega a^\dagger a + g \sum_i^2 (a\sigma_+^{(i)} + a^\dagger \sigma_-^{(i)}), \tag{1}$$

where $\sigma_z^{(i)}$, $\sigma_\pm^{(i)}$ ($i = 1, 2$) are atomic operators, ω_0 is atomic transition frequency, g is the coupling constant of individual atom to cavity field and a ($a^\dagger$) is the annihilation (creation) operator of cavity field with frequency ω. The generation of entangled state in the system (1) in laboratory has been implemented [12]. Various modifications and generalizations of the system (1) have been studied for preparing entangled states or realizing various kinds of quantum information processes [13, 14, 15, 16]. It is assumed that the cavity field are prepared initially in vacuum state $|0\rangle$, and the atom 1 is prepared in the mixed state $\lambda|e\rangle\langle e| + (1 - \lambda)|g\rangle\langle g|$ and the atom 2 is in the ground state $|g\rangle$, i.e.,

$$\rho(0) = |0\rangle\langle 0| \otimes [\lambda|e\rangle\langle e| + (1 - \lambda)|g\rangle\langle g|] \otimes |g\rangle\langle g|. \tag{2}$$

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The time evolution of $\rho(t)$ can be derived as follows,

$$
\begin{aligned}
\rho(t) = &\ \frac{\lambda}{8}[1 + \frac{\Delta^2}{\Omega^2} + (1 - \frac{\Delta^2}{\Omega^2})\cos\Omega t]|0\rangle\langle 0| \otimes |B_+\rangle\langle B_+| \\
&+ \frac{g^2\lambda}{\Omega^2}[1 - \cos\Omega t]|1\rangle\langle 1| \otimes |gg\rangle\langle gg| + \frac{\lambda}{4}|0\rangle\langle 0| \otimes |B_-\rangle\langle B_-| \\
&+ \frac{\sqrt{2}g\lambda}{2\Omega}[\frac{\Delta}{\Omega}(1 - \cos\Omega t) + i\sin\Omega t]|0\rangle\langle 1| \otimes |B_+\rangle\langle gg| \\
&+ \frac{\sqrt{2}g\lambda}{2\Omega}[\exp(\frac{i\Omega t - i\Delta t}{2}) - \exp(\frac{-i\Omega t - i\Delta t}{2})]|0\rangle\langle 1| \otimes |B_-\rangle\langle gg| \\
&+ \frac{\lambda}{4}\{(1 - \frac{\Delta}{\Omega})e^{i(\Omega+\Delta)t/2} + (1 + \frac{\Delta}{\Omega})e^{-i(\Omega-\Delta)t/2}\}|0\rangle\langle 0| \otimes |B_+\rangle\langle B_-| \\
&+ \frac{1-\lambda}{2}|0\rangle\langle 0| \otimes |gg\rangle\langle gg| + h.c.,
\end{aligned}
\tag{3}
$$

where $\Delta = \omega_0 - \omega$ is the detuning between the atoms and cavity field, $\Omega = (\Delta^2 + 8g^2)^{1/2}$, and $|B_\pm\rangle = \frac{\sqrt{2}}{2}(|eg\rangle \pm |ge\rangle)$ are the Bell states. By tracing out the degree of freedom of the cavity field, we obtain the reduced density matrix $\rho_s(t)$ describing the subsystem containing only two atoms,

$$
\begin{aligned}
\rho_s(t) = &\ \frac{\lambda}{8}[1 + \frac{\Delta^2}{\Omega^2} + (1 - \frac{\Delta^2}{\Omega^2})\cos\Omega t]|B_+\rangle\langle B_+| \\
&+ \frac{\lambda}{4}|B_-\rangle\langle B_-| + \{\frac{g^2\lambda}{\Omega^2}[1 - \cos\Omega t] + \frac{1-\lambda}{2}\}|gg\rangle\langle gg| \\
&+ \frac{\lambda}{4}\{(1 - \frac{\Delta}{\Omega})e^{i(\Omega+\Delta)t/2} + (1 + \frac{\Delta}{\Omega})e^{-i(\Omega-\Delta)t/2}\}|B_+\rangle\langle B_-| + h.c..
\end{aligned}
\tag{4}
$$

Firstly, we analyze the feasibility of preparing maximally entangled mixed states of two atoms in this cavity QED system. The concurrence [17] is adopted to quantify the bipartite entanglement between two atoms, and the linear entropy defined by $M = \frac{4}{3}(1 - \text{Tr}\rho_s^2)$ of the reduced density matrix is used to quantify the mixedness. In the situation with a rational value of $\frac{\Delta}{\Omega}$, the evolving density matrix is periodic. In the case with an irrational value of $\frac{\Delta}{\Omega}$, the evolving state is not periodic. The explicit analytical expression of the concurrence $C_s(t)$ characterizing the entanglement in $\rho_s(t)$ can be obtained as

$$
\begin{aligned}
C_s(t) = &\ \lambda(A^2 + B^2)^{\frac{1}{2}}, \\
A = &\ \frac{\Delta^2}{4\Omega^2} - \frac{1}{4} + \frac{1}{4}(1 - \frac{\Delta^2}{\Omega^2})\cos\Omega t, \\
B = &\ \frac{1}{2}(1 - \frac{\Delta}{\Omega})\sin(\Omega + \Delta)t/2 - \frac{1}{2}(1 + \frac{\Delta}{\Omega})\sin(\Omega - \Delta)t/2.
\end{aligned}
\tag{5}
$$

In the case with $\lambda = 1$, two atoms can be in the pure states at some specific times. The entanglement characterized by concurrence of those pure states are given by $C = |\sin\frac{\Delta k\pi}{\Omega}|$ which are achieved at the discrete times denoted by $t = 2k\pi/\Omega$ ($k = 1, 2, ...$). If $\frac{\Delta}{\Omega}$ is a rational number, the series $|\sin\frac{\Delta k\pi}{\Omega}|$ ($k = 1, 2, ...$) have finite and discrete values. While for the case that $\frac{\Delta}{\Omega}$ is an irrational number, the series $|\sin\frac{\Delta k\pi}{\Omega}|$ ($k = 1, 2, ...$) have infinite numbers of values, and this series can very approximately approach to any values between 0 and 1 according to Hurwitz's theorem in number theory. It means pure two-qubit states with any desired degree of entanglement can be very approximately generated for those cases with the irrational values of $\frac{\Delta}{\Omega}$.

In the large detuning limit, i.e., $g/|\Delta| \ll 1$, the population of the single mode cavity field will be very small in the time evolution, which leads very small entanglement between the atoms and the cavity field. Therefore the mixedness of the subsystem containing two atoms is very small. In the small detuning limit, i.e. $|\Delta|/g \ll 1$ but not zero, and simultaneously $\frac{\Delta}{\Omega}$ is an irrational number, the trajectories of the reduced density operator of two atoms in the concurrence versus linear-entropy plane exhibit a kind of "quasi-ergodic" property, roughly speaking, where "quasi-ergodic" means there is no distinct interspaces in the pattern formed by the trajectory of the evolving state in the concurrence versus linear-entropy plane.

In Fig.1, the concurrence versus mixedness of two atoms are depicted for different values of detuning. In the resonant case, the concurrence of two atoms increases (decreases) with the increase (decrease) of mixedness of their reduced density matrix. In the resonant situation, the evolving reduced density matrix $\rho_s(t)$ in Eq.(4) can not become any one of the maximally entangled mixed state in the plane of linear entropy-concurrence. Interestingly, in the off-resonant
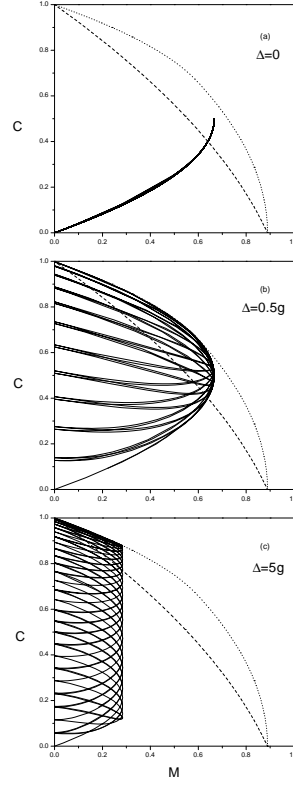
FIG. 1: The concurrence versus mixedness of two atoms are depicted. The trajectory is chosen from the scaled time $gt \in [0, 50]$. The dash line and dot line in (a), (b) and (c) represent the Werner state and the maximally entangled mixed state (the frontier of the concurrence versus linear entropy) respectively. (a) $\Delta = 0$; (b) $\Delta = 0.5g$; (c) $\Delta = 5g$. In three cases, $\lambda = 1$.

case, part of the frontier of the concurrence versus linear entropy can be very approximately reached by the evolving reduced density matrix of two atoms. However, the region in the frontier which can be approximately approached by the evolving reduced density matrix reduces with the increase of the detuning. Approximately, two atoms can acquire desired pure state concurrence between 1 and 0 for both the small detuning case and large detuning case in the precondition that $\frac{\Delta}{\Omega}$ is an irrational number.

From Fig.2, we can understand the influence of initial mixedness of two atoms on the entanglement and mixedness of their evolving reduce density matrix. It is shown that the range of the frontier which can be approximately approached is reduced when the initial mixedness of the atoms increases. For the case with $\lambda = 0.6$, two atoms can evolve into a state with smaller mixedness than their initial state which is different from other cases with $\lambda = 0.9$ and $\lambda = 0.7$. One can also find that the patterns formed by the trajectories are mirror symmetric with the horizontal axis labeled by the half of the concurrence of maximally entangled mixed state corresponding to the initial linear entropy.

Bell's inequality test with entangled atoms inside a cavity have been extensively studied [18]. The most commonly discussed Bell inequality is the CHSH inequality [19, 20]. The CHSH operator reads

$$\hat{B} = \vec{a} \cdot \vec{\sigma} \otimes (\vec{b} + \vec{b'}) \cdot \vec{\sigma} + \vec{a'} \cdot \vec{\sigma} \otimes (\vec{b} - \vec{b'}) \cdot \vec{\sigma}, \tag{6}$$

where $\vec{a}, \vec{a'}, \vec{b}, \vec{b'}$ are unit vectors. In the above notation, the Bell inequality reads

$$|\langle \hat{B} \rangle| \leq 2. \tag{7}$$

The maximal amount of Bell violation of a state ρ is given by [21]

$$|\mathcal{B}|_{max} = 2\sqrt{\kappa + \tilde{\kappa}}, \tag{8}$$

where κ and $\tilde{\kappa}$ are the two largest eigenvalues of $T_\rho^\dagger T_\rho$. The matrix T_ρ is determined completely by the correlation functions being a 3×3 matrix whose elements are $(T_\rho)_{nm} = \text{Tr}(\rho \sigma_n \otimes \sigma_m)$. Here, $\sigma_1 \equiv \sigma_x$, $\sigma_2 \equiv \sigma_y$, and $\sigma_3 \equiv \sigma_z$ denote the usual Pauli matrices. The quantity $|\mathcal{B}|_{max}$ is called as the maximal violation measure, which indicates the
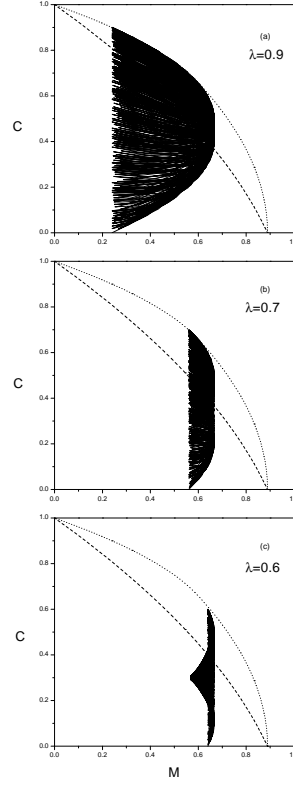
FIG. 2: The concurrence versus mixedness of two atoms are displayed in the cases in which one of the atoms is initially in three different mixed states: (a) $\lambda = 0.9$; (b) $\lambda = 0.7$; (c) $\lambda = 0.6$. The trajectories are also chosen from the scaled time $gt \in [0, 500]$. The dash line and dot line in (a), (b) and (c) represent the Werner state and the maximally entangled mixed state respectively. It is shown that two atoms can approximately approach part of the maximally entangled mixed states though the range of the frontier which can be approached is reduced when the initial mixedness of the atom increases. In three cases, $\Delta = 0.5g$.

Bell violation when $|\mathcal{B}|_{max} > 2$ and the maximal violation when $|\mathcal{B}|_{max} = 2\sqrt{2}$. For the density operator ρ_s in Eq.(4), $\kappa + \tilde{\kappa}$ can be written as follows

$$\kappa + \tilde{\kappa} = \varsigma + \max[\varsigma, \zeta], \tag{9}$$

where

$$
\begin{aligned}
\varsigma &= \frac{4g^4}{\Omega^4}(1 - \cos\Omega t)^2 \\
&+ \frac{1}{4}[(1 - \frac{\Delta}{\Omega})\sin\frac{(\Omega + \Delta)t}{2} - (1 + \frac{\Delta}{\Omega})\sin\frac{(\Omega - \Delta)t}{2}]^2 \\
\zeta &= (\frac{\Delta^2 + 4g^2}{\Omega^2} + \frac{4g^2}{\Omega^2}\cos\Omega t)^2
\end{aligned}
\tag{10}
$$

in the case with $\lambda = 1$. In Ref.[7], the analytical form of the mixed states which possess the maximal value of $|\mathcal{B}|_{max}$ of two qubits for a given linear entropy has been derived. Here, part of the frontier of the maximal Bell violation versus the linear entropy can also be very approximately approached by the evolving state of two atoms (see Fig.3b). In Fig.3a, our calculations show that two atoms can not violate the Bell-CHSH inequality in the resonant case, though two atoms could get entangled. While in the off-resonant case, the bell violation of the atom 1 and the atom 2 can emerge in their long-time evolution, even though the detuning Δ is very very small.

If the pure phase decoherence mechanism is considered, the master equation governing the time evolution of the system under the Markovian approximation is given by [22, 23]

$$\frac{d\rho}{dt} = -i[H, \rho] - \frac{\gamma}{2}[H, [H, \rho]], \tag{11}$$

where γ is the phase decoherence rate. The explicit analytical expression of the concurrence $C_\gamma(t)$ characterizing the
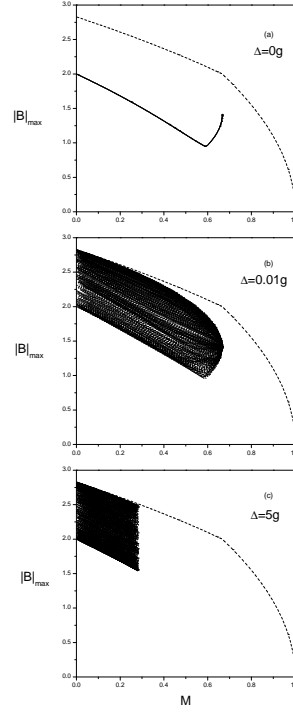
FIG. 3: The maximal Bell violation $|B|_{max}$ versus the mixedness of two atoms is displayed for three different values of the detuning: (a) $\Delta = 0$; (b) $\Delta = 0.01g$; (c) $\Delta = 5g$. The trajectory is chosen from the scaled time $gt \in [0, 500]$. The dash line represents the frontier of maximal Bell violation versus the linear entropy, namely, for a given linear entropy, the maximal value of $|B|_{max}$ of two atoms can not exceed the dash line. In three cases, $\lambda = 1$.

entanglement of two atoms in the presence of phase decoherence can be obtained as

$$
\begin{aligned}
C_\gamma(t) &= \lambda(A_\gamma^2 + B_\gamma^2)^{\frac{1}{2}}, \\
A_\gamma &= \frac{\Delta^2}{4\Omega^2} - \frac{1}{4} + \frac{1}{4}(1 - \frac{\Delta^2}{\Omega^2})\cos\Omega t \exp(-\frac{\gamma t}{2}\Omega^2), \\
B_\gamma &= \frac{1}{2}(1 - \frac{\Delta}{\Omega})\sin(\Omega + \Delta)t/2\exp[-\frac{\gamma t}{8}(\Omega + \Delta)^2] - \frac{1}{2}(1 + \frac{\Delta}{\Omega})\sin(\Omega - \Delta)t/2\exp[-\frac{\gamma t}{8}(\Omega - \Delta)^2],
\end{aligned} \tag{12}
$$

if the system is initially in the same state as $\rho(0)$ in Eq.(2). From Eq.(12), we can easily know that the phase decoherence does not completely destroy the entanglement but generate a stationary entangled state of two atoms. The concurrence $C_\gamma(t)$ is not larger that 0.5 in the resonant case. The entanglement of stationary state decreases with the increase of the detuning. The phase decoherence changes trajectories in the plane of concurrence versus linear entropy of the evolving state and makes the trajectories become chaotic. In Fig.4, we display the concurrence versus mixedness of two atoms in the presence of phase decoherence. The evolving reduced density matrix of two atoms can approximately approach to wider region of the maximally entangled mixed states, if both the ratio Δ/g and the decoherence rate γ are appropriate.

In summary, we have investigated a possible scheme for generating the maximally entangled mixed state of two atoms which are symmetrically coupled to a single mode optical cavity field. It is shown that two atoms can not achieve the maximally entangled mixed state in the resonant case. In the off-resonant case, the reduced density matrix of two atoms can approximately approach to the maximally entangled state in their evolution. The distinct roles of the rational values or irrational values of $\frac{\Delta}{\Omega}$ in the long-time behaviors of entanglement and mixedness of two atoms have been clarified. The influence of the phase decoherence and the initial mixedness of the atoms is also discussed. These results presented here maybe have potential applications in the domain of quantum information and quantum
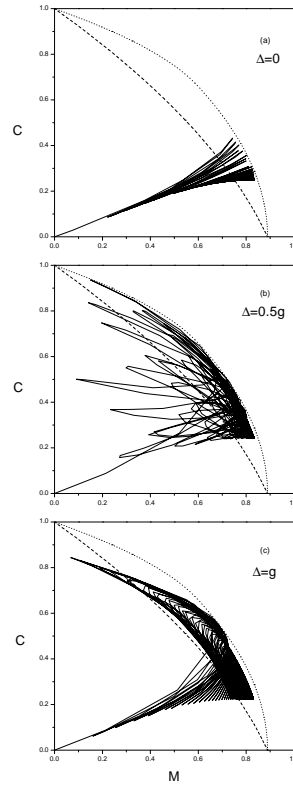
FIG. 4: We display the concurrence versus mixedness of two atoms in the presence of phase decoherence. The trajectories are chosen from the scaled time $gt \in [0, 500]$. The dash line and dot line in (a), (b) and (c) represent the Werner state and the maximally entangled mixed state, respectively. (a) $\Delta = 0$; (b) $\Delta = 0.5g$, it can be observed that an appropriate detuning and decoherence rate can make two atoms possess of ability to approach the wider region of the frontier; (c) $\Delta = g$. In three cases, $\gamma = 0.01/g$ and $\lambda = 1$.

communication and in the field dealing with the fundamental tests of quantum mechanics.

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